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# A LOWER BOUND FOR $v$ IN A $t-(v, k, \lambda)$ DESIGN

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In this paper it is shown that if $v \geq k+1$ then $v \geq t-1+(k-t+1)(k-t+2)/\lambda$, where $v$, $k$, $\lambda$ and $t$ are the characteristic parameters of a $t-(v, k, \lambda)$ design. We compare this bound with the known lower bounds on $v$.

Let $X$ be a finite set and let $\mathcal{A}$ be a collection of subsets of $X$, called blocks. $(X, \mathcal{A})$ is said to be a $t-(v, k, \lambda)$ design iff

(1) (1) $|X| = v$,
(2) $|A| = k$, for all $A \subset \mathcal{A}$,
(3) every $t$-subset of $X$ is contained in exactly $\lambda$ members of $\mathcal{A}$

Since there are $\binom{v}{t}$ $t$-subsets, each of which occurs $\lambda$ times, and since every block contains $\binom{k}{t}$ $t$-subsets, it follows that

$$(2) \qquad b = \lambda \binom{v}{t} \Big/ \binom{k}{t},$$

where $b$ is the cardinality of $\mathcal{A}$.

Let $(X, \mathcal{A})$ be a $t-(v, k, \lambda)$ design. It follows from definition (1) that $(X, \mathcal{A})$ is also an $i-(v, k, \lambda_i)$ design for $1 \leq i \leq t-1$. Let $I$ be an arbitrary $i$-subset of $X$. Since there are $\binom{v-i}{t-i}$ $t$-subsets, containing $I$, each of which occurs $\lambda$ times, and since every block, containing $I$, includes $\binom{k-i}{t-i}$ $t$-subsets, containing $I$, it follows that

$$(3) \qquad \lambda_i = \lambda \binom{v-i}{t-i} \Big/ \binom{k-i}{t-i}, \qquad i = 1, 2, \ldots, t-1.$$

This proves a well known necessary condition (cf. [1]) for the existence of a $t-(v, k, \lambda)$ design, namely

(4) $$\lambda\binom{v-i}{t-i}\Big/\binom{k-i}{t-i} \text{ is an integer for } i = 0, 1, ..., t-1 .$$

Known are some inequalities, which have to be satisfied for the existence of a $t-(v, k, \lambda)$ design.

For $t = 2$, Fisher showed that

(5) $$v \geqslant k+1 \quad \text{implies} \quad b \geqslant v .$$

In 1968, Petrenjuk showed [2] that for $t = 4$,

(6) $$v \geqslant k+2 \quad \text{implies} \quad b \geqslant \binom{v}{2} .$$

In 1971, Wilson and Ray-Chaudhuri [4] generalized (5) and (6) by showing

(7) $$v \geqslant k + [\tfrac{1}{2}t] \quad \text{implies} \quad b \geqslant \binom{v}{[\frac{1}{2}t]} .$$

In 1971, Petrenjuk showed [3] that in a $-(v, k, \lambda)$ design with $\lambda = 1$ and $1 < t < k < v-t$,

(8) $$\binom{v-1}{t-1} \geqslant \binom{k}{t-1}\binom{k-1}{t-1} .$$

We shall now state a new inequality.

**Theorem 1.** *In a $t-(v, k, \lambda)$ design $(X, \mathcal{A})$ with $v \geqslant k+1$,*

(9) $$v \geqslant t-1+\frac{(k-t+1)(k-t+2)}{\lambda} .$$

**Proof.** Let $Y$ be a $(t-2)$-subset of $X$ and let $Z = X\setminus Y$. Let $\mathcal{B} := \{A \in \mathcal{A} : Y \subset A\}$ and let $\mathcal{C} := \{B\setminus Y : B \in \mathcal{B}\}$

It can easily be verified from the definitions that $(Z, \mathcal{C})$ is a $2-(v-t+2, k-t+2, \lambda)$ design. The number of blocks in $(Z, \mathcal{C})$ equals $|\mathcal{C}| = |\mathcal{B}| = \lambda_{t-2}$, hence (5) implies $\lambda_{t-2} = |\mathcal{C}| \geqslant |Z| = v-t+2$.

Since

$$\lambda_{t-2} = \lambda\frac{(v-t+2)(v-t+1)}{(k-t+2)(k-t+1)}$$

by (3), we have shown

$$v - t + 1 \geqslant \frac{(k-t+2)(k-t+1)}{\lambda}\,.$$

In order to compare this inequality with (7) and (8) we substitute (3) in (7). Now (7) can be restated as

$$\text{(10)}\qquad (v - [\tfrac{1}{2}t])(v - [\tfrac{1}{2}t] - 1)\dots(v - t + 1) \geqslant \frac{k(k-1)\dots(k-t+1)}{\lambda([\frac{1}{2}t])!}\,.$$

Defining $P(k)$, $W(k)$ and $T(k)$ as the smallest $v$ satisfying (8), (10), resp. Theorem 1, for fixed $t$ and $\lambda$, it is now easy to find the asymptotic behavior of these entities (remark that $P(k)$ is only defined for $\lambda = 1$).

$$P(k) \sim \frac{k^2}{((t-1)!)^{1/(t-1)}} \qquad (k \to \infty)\,,$$

$$W(k) \sim \frac{k^{(2n+1)/(n+1)}}{(\lambda \cdot n!)^{1/(n+1)}} \qquad \text{for } t = 2n+1 \qquad (k \to \infty)\,,$$

$$\sim \frac{k^2}{(\lambda \cdot n!)^{1/n}} \qquad \text{for } t = 2n \qquad (k \to \infty)\,,$$

$$T(k) \sim \frac{k^2}{\lambda} \qquad (k \to \infty)\,.$$

Hence

$$\lim_{k\to\infty} \frac{T(k)}{P(k)} = ((t-1)!)^{1/(t-1)} > 1 \qquad \text{for } t \geqslant 3\,,$$

$$\lim_{k\to\infty} \frac{T(k)}{W(k)} = \infty \qquad \text{if } t \text{ is odd}$$

and

$$\lim_{k\to\infty} \frac{T(k)}{W(k)} = \sqrt[n]{\frac{n!}{\lambda^{n-1}}} \qquad \text{if } t = 2n\,.$$

In this last case the limit is greater than 1 iff $n! > \lambda^{n-1}$.

## Note added in proof

After the submission of this article, the author found out by a private communication that Fisher et al. have proved the following generalisation, using similar techniques as above:

$$\lambda\binom{v-t+s}{s} \geqslant \binom{k-t+2s}{s}\binom{k-t+s}{s}, \qquad 0 \leqslant s \leqslant [\tfrac{1}{2}t] .$$

## References

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